Worcester County Mathematics League

Varsity Meet 4 – March 2, 2022

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Worcester County Mathematics League
Varsity Meet 4 - March 2, 2022
Round 1 - Elementary Number Theory

All answers must be in simplest exact form in the answer section.

1. Express the sum $1232_5 + 341_5$ in base five.

2. Find the sum of the least common multiple and the greatest common factor of the numbers 24, 36, and 60.

3. Find the number of ordered pairs of positive integers, (m, n), which satisfy the equation $(m \cdot n)^k = 1296$ for some integer k.

ANSWERS

(1 pt) 1. ________________

(2 pts) 2. ________________

(3 pts) 3. ________________

Douglass, Doherty, Worc Academy

Worcester County Mathematics League
Varsity Meet 4 - March 2, 2022
Round 2 - Algebra I

All answers must be in simplest exact form in the answer section.

1. Alan buys four new tires and a news pare tire for his car. He rotates his tires so that after driving 5,000 miles, every tire has been used for the same number of miles. For how many miles was each tire used?

2. Find the sum of all integers x for which $0 \leq (x-5)(x-7)(x-9) \leq 33$.

3. When the expression $(a+(b-2c))(a^2-a(b-2c)+(b-2c)^2)$ is expanded into a sum of product terms, find the product of the non-zero coefficients. For example, $-13a^2b$ is a product term with coefficient -13 and $6c^3$ is a product term with coefficient 6.

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. ____________________

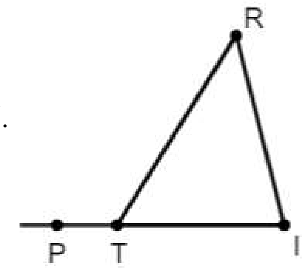
(3 pts) 3. ____________________

Burncoat, Marlborough, Shrewsbury

Worcester County Mathematics League
Varsity Meet 4 - March 2, 2022
Round 3 - Geometry

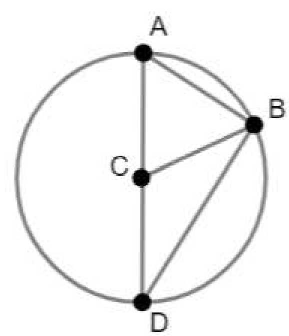
All answers must be in simplest exact form in the answer section.

1. In the figure shown at right, $m\angle PTR = 3m\angle R$ and $m\angle R = \frac{2}{3}m\angle RTI$. Find $m\angle I$ in degrees.



2. Circle P and circle Q (not shown) have a common chord of length 16. If the radius of circle P is 10 and the radius of circle Q is 17, find *all* possible values for PQ.

3. Circle C is shown at right with diameter $\overline{AD}$. Chord $\overline{DB}$ has length 8. If the area of ΔABC is 10, then the area of circle C can be expressed as a ratio of integers times π, or $\frac{m}{n}\pi$. Find the ordered pair (m, n) where m and n are integers whose greatest common factor is 1.



ANSWERS

(1 pt) 1. $m\angle I =$ ________________ $^\circ$

(2 pts) 2. $PQ =$ ________________

(3 pts) 3. $(m, n) =$ ________________

Tahanto, Bancroft, Shrewsbury

Worcester County Mathematics League
Varsity Meet 4 - March 2, 2022
Round 4 - Logs, Exponents, and Radicals

All answers must be in simplest exact form in the answer section.

1. Solve for x:

$$\log_{10} x + \log_{10} 4 = 2$$

2. If $N > 1$ in the equation below, find x.

$$\sqrt[2]{N\sqrt[3]{N\sqrt[3]{N}}} = N^x$$

3. Let $x = \dfrac{1}{\log_3(1001)} + \dfrac{1}{\log_5(1001)} + \dfrac{1}{\log_7(1001)} + \dfrac{1}{\log_9(1001)} + \dfrac{1}{\log_{11}(1001)} + \dfrac{1}{\log_{13}(1001)}$.
   If x is expressed as $1 + \log_b(M)$, find the ordered pair (b, M).

ANSWERS

(1 pt) 1. $x =$ ____________________

(2 pts) 2. $x =$ ____________________

(3 pts) 3. $(b, M) =$ ____________________

Algonquin, Bancroft, QSC

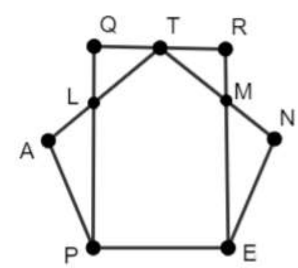
Worcester County Mathematics League
Varsity Meet 4 - March 2, 2022
Round 5 - Trigonometry

All answers must be in simplest exact form in the answer section.

1. Find $\csc\theta$ if $\cos\theta = \frac{\sqrt{3}}{2}$ and $\cot\theta = -\sqrt{3}$.

2. If $\sin A = -\frac{12}{13}$ and $\cos B = -\frac{3}{5}$, where $180° < A < 270°$ and $180° < B < 270°$, find $\cos(A+B)$. Express your result as a fraction in simplest terms.

3. In the figure shown at right, $PENTA$ is a regular pentagon that shares side $PE = 2$ with rectangle $PERQ$. L is the intersection of $\overline{AT}$ and $\overline{PQ}$. Likewise, M is the intersection of $\overline{TN}$ and $\overline{RE}$.
If $AL = \dfrac{a}{b + c\sin^2 18°}$, find (a, b, c).



ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. ____________________

(3 pts) 3. $(a, b, c) =$ ____________________

Notre Dame, Clinton, Quaboag

Worcester County Mathematics League
Varsity Meet 4 - March 2, 2022
Team Round

All answers must be in simplest exact form in the answer section.

1. Three ships travel continuously from New York to Le Havre, France and back. One ship makes the round trip in 12 days, another ship makes the round trip in 20 days, and the third ship makes the round trip in 28 days. If all three ships leave New York on the same day, how many days will elapse before all three ships first arrive in Le Havre on the same day? Assume that the one way trip times are the same for either direction, and that no time is spent in either port.

2. In Mr. Smith's math class, 36 students took a test. If the average passing grade was 78, the average failing grade was 60, the class average was 71, and the passing grade was 70, how many students passed the test?

3. Given that $ABCD$ is a trapezoid that has base angles of $\angle DAB$ and $\angle CBA$ measuring 45 degrees and 60 degrees respectively, $BC = 4$ and $DC = 5$, find the area of $ABCD$.

4. The lengths of the sides of a triangle are $\log_3 5$, $\log_3 11$, and $\log_3 x$. If m is the smallest integer value for x, and n is the largest integer value for x, find (m, n).

5. In triangle $\triangle ABC$, $AB = 8$, $AC = 20$, and $\tan \angle A = \frac{7}{5}$. Find the area of $\triangle ABC$ and express your answer in simplest radical form.

6. Find the polynomial with leading coefficient 1 of lowest order that is a common multiple of $x^2 - 25$, $x^3 - 125$, and $x^2 - 10x + 25$. Provide an answer that is completely factored over the rationals.

7. Find all solutions (x, y) to the following system of equations:

$$\frac{x}{4} + \frac{y}{3} = \frac{9}{2}$$

$$\frac{4}{x} + \frac{3}{y} = 1$$

8. Let r be the result of doubling both the base and the exponent of the expression a^b, where both $a > 0$ and $b > 0$. If r is equal to the product of a^b and x^b, find x in terms of a and b.

9. If $\tan 2x = \frac{24}{7}$ and $90° < x < 180°$, find $\sec x$.

Burncoat, St. John's, Hudson, St. John's, Auburn, Auburn, Hudson, Burncoat, Westboro

Worcester County Mathematics League
Varsity Meet 4 - March 2, 2022
Team Round Answer Sheet

ANSWERS

1. ____________________ days

2. ____________________ students

3. ____________________

4. $(m, n) =$ ________________

5. ____________________

6. ____________________

7. { __________________ }

8. $x =$ __________________

9. ____________________

Burncoat, St. John's, Hudson, St. John's, Auburn, Auburn, Hudson, Burncoat, Westboro

Worcester County Mathematics League

Varsity Meet 4 - March 2, 2022

Answer Key

Round 1 - Elementary Number Theory

1. 2123 or 2123_5
2. 372
3. 38

Round 2 - Algebra I

1. 4000
2. 37
3. 576

Round 3 - Geometry

1. 80
2. $\{9, 21\}$ any order, with or without { }
3. $(89, 4)$ exact order

Round 4 - Logs, Exponents, and Radicals

1. 25
2. $\frac{13}{18}$
3. $(1001, 135)$ exact order

Round 5 - Trigonometry

1. -2
2. $-\frac{33}{65}$
3. $(2, 3, -4)$ exact order

Team Round

1. 210
2. 22
3. $6 + 12\sqrt{3}$
4. $(3, 54)$ exact order
5. $\frac{280\sqrt{74}}{37}$
6. $(x-5)^2(x+5)(x^2+5x+25)$ must be factored, but factors can be in any order
7. $(6, 9), \left(12, \frac{9}{2}\right)$

   or

   $\left(12, \frac{9}{2}\right), (6, 9)$
8. $4a$
9. $-\frac{5}{3}$

Round 1 - Elementary Number Theory

1. Express the sum $1232_5 + 341_5$ in base five.

Solution: The sum is shown below, in tabular form:

$$\begin{array}{rl} & 1\cdot 125 + 2\cdot 25 + 3\cdot 5 + 2\cdot 1 \\ & \quad + 3\cdot 25 + 4\cdot 5 + 1\cdot 1 \\ \hline & = 1\cdot 125 + 5\cdot 25 + 7\cdot 5 + 3\cdot 1 \\ & = 1\cdot 125 + 6\cdot 25 + 2\cdot 5 + 3\cdot 1 \\ & = 2\cdot 125 + 1\cdot 25 + 2\cdot 5 + 3\cdot 1 \\ & = \boxed{2123}_5 \end{array}$$

2. Find the sum of the least common multiple and the greatest common factor of the numbers 24, 36, and 60.

Solution: The prime factorizations of 24, 36, and 60 are $2^3 \cdot 3^1 \cdot 5^0$, $2^2 \cdot 3^2 \cdot 5^0$ and $2^2 \cdot 3^1 \cdot 5^1$. The greatest common factor (GCF) of these three numbers is equal to the product of 2, 3, and 5, each taken to the lowest power found in the three factorization. Thus, GCF(24,26,60) $= 2^2 \cdot 3^1 \cdot 5^0 = 12$.

The Least Common Multiple (LCM) of the three numbers is equal to the product of 2, 3, and 5, each taken to the highest power found in the three factorizations. Thus, LCM(24,36,60) $= 2^3 \cdot 3^2 \cdot 5^1 = 360$. Adding the LCM to the GCF yields $12 + 360 = \boxed{372}$.

3. Find the number of ordered pairs of positive integers, (m, n), which satisfy the equation $(m \cdot n)^k = 1296$ for some integer k.

Solution: Note that 1296 is divisible by 6, and that $1296/6 = 216 = 6^3$, so $1296 = 6^4$. Since m and n are positive integers, and k is an integer, then $k = 4, 2$ or 1. When $k = 4$, then $m \cdot n = 6 = 2 \cdot 3$. When $k = 2$, then $m \cdot n = 36 = 2^2 \cdot 3^2$. Finally, when $k = 1$, then $m \cdot n = 1296 = 2^4 * 3^4$.

Two facts help organize the counting. First, the number of ordered pairs of integers (m, n) where $m \cdot n = k$ is equal to the number of positive factors of k. This statement is true because m and n must be factors of k, and for each m, $n = k \div m$. Next, the number whose prime factorization is given by $p^a \cdot q^b$ has exactly $(a + 1)(b + 1)$ factors. This is because the prime p could divide a factor anywhere from 0 to a times, a total of $a + 1$ possibilities. A similar result holds for prime q. Thus there are $(1 + 1)(1 + 1) = 4$ ordered pairs of solutions when $k = 4$, $(2 + 1)(2 + 1) = 9$ ordered pairs of solutions when $k = 2$, and $(4 + 1)(4 + 1) = 25$ ordered pairs of solutions when $k = 1$, for a total of $4 + 9 + 25 = \boxed{38}$ ordered pairs

Round 2 - Algebra I

1. Alan buys four new tires and a new spare tire for his car. He rotates his tires so that after driving 5,000 miles, every tire has been used for the same number of miles. For how many miles was each tire used?

Solution: For each mile the car is driven, four tires are used. After 5000 miles are driven, the group of tires has been used for $4 \cdot 5000 = 20000$ tire-miles in total. Since each of the 5 tires share that load equally, each tire is used for $20000/5 = \boxed{4000}$ miles.

2. Find the sum of all integers x for which $0 \leq (x-5)(x-7)(x-9) \leq 33$.

Solution: This inequality can be solved by cases.

- When $x = 5, 7,$ or 9, one of the three terms ($x-5$, $x-7$, or $x-9$) equals 0 so that the product is 0 and the inequality is satisfied.
- If $x < 5$ then all three terms are negative, and the product is negative. Therefore there are no solutions less than 5.
- If $5 < x < 7$ then two terms are negative and one is positive, and the product is positive. In this case, $x = 6$ and the product $(1)(-1)(-3) = 3$ satisfies the inequality
- If $7 < x < 9$ then two terms are positive and one is negative, the product is negative, and there are no solutions in that interval.
- Finally, if $x > 9$ then all terms are positive and the product is positive. When $x = 10$ the product is $(5)(3)(1) = 15$, satisfying the inequality. When $x = 11$ the product $(6)(4)(2) = 48$ doesn't satisfy the inequality. After $x = 11$, the product only increases.

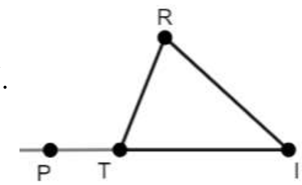
Thus, the only values that satisfy the inequality are 5, 6, 7, 9, and 10, and $5+6+7+9+10 = \boxed{37}$.

3. When the expression $(a+(b-2c))(a^2-a(b-2c)+(b-2c)^2)$ is expanded into a sum of product terms, find the product of the non-zero coefficients. For example, $-13a^2b$ is a product term with coefficient -13 and $6c^3$ is a product term with coefficient 6.

Solution: Let $b-2c = d$. Then the expression can be rewritten as the product $(a+d)(a^2-ad+d^2)$, which is the factored form of a^3-d^3. Now the original expression can be expanded using the identity for the cube of a binomial: $a^3-(b-2c)^3 = a^3-(b^3-3b^2(2c)+3b(2c)^2-(2c)^3) = a^3-b^3+6b^2c-12bc^2+8c^3$. The non-zero coefficients are 1, -1, 6, -12, and 8 and $1(-1)(6)(-12)(8) = 6 \cdot 12 \cdot 8 = \boxed{576}$.

Round 3 - Geometry

1. In the figure shown at right, $m\angle PTR = 3m\angle R$ and $m\angle R = \frac{2}{3}m\angle RTI$. Find $m\angle I$ in degrees.

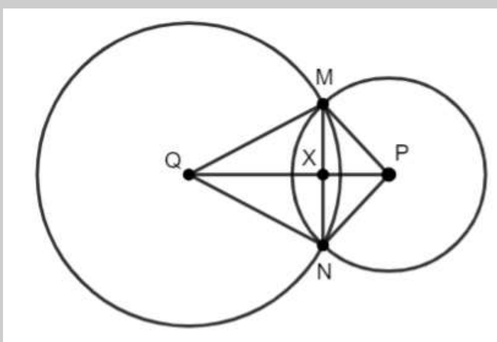


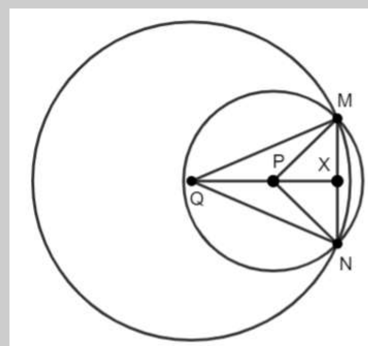
Solution: Let $m\angle R = y$. Applying the given information, $m\angle PTR = 3y$ and $y = \frac{2}{3}m\angle RTI$, or $m\angle RTI = \frac{3}{2}y$. Note that $\angle PTR$ and $\angle RTI$ form a line, so $180° = m\angle PTR + m\angle RTI = 3y + \frac{3}{2}y = \frac{9}{2}y$. Solving for y, $y = \frac{2\cdot 180}{9} = 2\cdot 20 = 40°$, and $m\angle PTR = 3\cdot 40 = 120°$.

By the exterior angle theorem (the measure of an exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles),$m\angle PTR = m\angle I + m\angle R$, or $120 = x + 40$. Thus $x = 120 - 40 = \boxed{80}^{\circ}$.

2. Circle P and circle Q (not shown) have a common chord of length 16. If the radius of circle P is 10 and the radius of circle Q is 17, find *all* possible values for PQ.

Solution: There are two cases where circles P and Q share a common chord. As shown below, the center of circle P may be outside or inside circle Q.



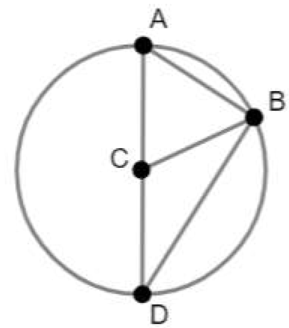


The quadrilateral $QMPN$ in the figure on the right is a kite, with two pairs of congruent adjacent sides ($QM = QN = 17$ and $PM = PN = 10$). A property of kites is that its diagonals are perpendicular. Furthermore, one diagonal, $\overline{PQ}$ in $QMPN$, divides the kite into congruent triangles and also bisects the other diagonal. Thus, the common chord $\overline{MN}$, as labeled in both figures, is perpendicular to $\overline{PQ}$ and is bisected by $\overline{PQ}$. The diagonals intersect at X, as labeled in the diagram, so triangles ΔQXM and ΔPXM are right triangles. Also, $MX = XN = MN/2 = 8$.

Now the Pythagorean Theorem can be applied to find QX and PX:

$$QX^2 = QM^2 - MX^2 = 17^2 - 8^2 = 289 - 64 = 225$$
$$PX^2 = PM^2 - MX^2 = 10^2 - 8^2 = 100 - 64 = 36$$

Taking square roots, $QX = 15$ and $PX = 6$. Note that the triangles ΔQXM and ΔPXM have the same dimensions in both figures, so these values apply to both cases. In the left figure $PQ = QX + PX$, and in the right figure, $PQ = QX - PX$. Thus, the two possible values for PQ are $15+6$ and $15-6$, or $\boxed{\{21, 9\}}$.



3. Circle C is shown at right with diameter $\overline{AD}$. Chord $\overline{DB}$ has length 8. If the area of ΔABC is 10, then the area of circle C can be expressed as a ratio of integers times π, or $\frac{m}{n}\pi$. Find the ordered pair (m, n) where m and n are integers whose greatest common factor is 1.

Solution: Because $\overline{AD}$ is a diameter, ΔABD is inscribed in a semicircle, and is therefore a right triangle with hypotenuse $\overline{AD}$. This statement is true because $\angle ABD$ is an inscribed angle whose measure is half the measure of the semicircle that it intercepts, so its measure is $180 \div 2 = 90°$. Since C is the center of the circle, it is the midpoint of $\overline{AD}$, and $\overline{BC}$ is a median of ΔABD. A median of any triangle divides the triangle into two triangles of equal areas, that is, half the area of the original triangle. Therefore the area of ΔABD is twice the area of ΔABC and is equal to 20.

The area of a right triangle is equal to half the product of its two legs. Thus the area of ΔABD, $[\Delta ABD] = 20 = \frac{AB \cdot DB}{2}$, and $AB = \frac{2 \cdot 20}{8} = 5$ after substituting the given value $DB = 8$. The diameter of circle C is found by applying the Pythagorean Theorem: $AD^2 = AB^2 + BD^2 = 5^2 + 8^2 = 25 + 64 = 89$. The area of a circle is equal to $\pi r^2 = \pi \frac{d^2}{4}$ where r is the radius of the circle and $d = 2r$ is the diameter of the circle. Therefore the area of circle C is equal to $\frac{89}{4}\pi$ and $(m, n) = \boxed{(89, 4)}$.

Round 4 - Logs, Exponents, and Radicals

1. Solve for x:

$$\log_{10} x + \log_{10} 4 = 2$$

.

Solution: The left hand side of the expression can be simplified into a single logarithm using the product law of logarithms ($\log m \cdot n = \log m + \log n$):

$$\log_{10} x + \log_{10} 4 = \log_{10} x \cdot 4 = \log_{10} 4x = 2$$

Now, applying the inverse relationship of logarithms and exponentials ($\log_b x = n \Leftrightarrow b^n = x$):

$$10^2 = 4x$$

or $4x = 100$, so $x = \frac{100}{4} = \boxed{25}$.

2. If $N > 1$ in the equation below, find x.

$$\sqrt[2]{N\sqrt[3]{N\sqrt[3]{N}}} = N^x$$

Solution: Note that $\sqrt[k]{x}$ is equal to a fractional exponential of x, specifically $\sqrt[k]{x} = (x)^{\frac{1}{k}}$. Apply that property, together with the laws of exponents $(b^m)^n = b^{(m \cdot n)}$ and $b \cdot b^x = b^{1+x}$, to convert the left hand side of the equation to a simple exponential with base N:

$$\begin{aligned}\sqrt[2]{N\sqrt[3]{N\sqrt[3]{N}}} &= \sqrt[2]{N\sqrt[3]{N \cdot N^{\frac{1}{3}}}} = \sqrt[2]{N\sqrt[3]{N^{\frac{4}{3}}}} \\ &= \sqrt[2]{N \cdot \left(N^{\frac{4}{3}}\right)^{\frac{1}{3}}} = \sqrt[2]{N \cdot N^{\frac{4}{9}}} \\ &= \sqrt[2]{N^{\frac{13}{9}}} = \left(N^{\frac{13}{9}}\right)^{\frac{1}{2}} = N^{\frac{13}{18}} = N^x\end{aligned}$$

Finally, if $b^x = b^y$ then $x = y$. Therefore $x = \boxed{\frac{13}{18}}$.

3. Let $x = \dfrac{1}{\log_3(1001)} + \dfrac{1}{\log_5(1001)} + \dfrac{1}{\log_7(1001)} + \dfrac{1}{\log_9(1001)} + \dfrac{1}{\log_{11}(1001)} + \dfrac{1}{\log_{13}(1001)}$.
   If x is expressed as $1 + \log_b(M)$, find the ordered pair (b, M).

Solution: First change the base of each of the logarithms using the identity $\log_b a = \frac{1}{\log_a b}$ to obtain an equivalent expression for x:

$$x = \log_{1001} 3 + \log_{1001} 5 + \log_{1001} 7 + \log_{1001} 9 + \log_{1001} 11 + \log_{1001} 13$$

Next, combine the sum of logarithms into a single logarithm using the product rule of logarithms:

$$x = \log_{1001} 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13$$

Note that $1001 = 7 \cdot 11 \cdot 13$ and that $\log_b b = 1$, so:

$$\begin{aligned} x &= \log_{1001} (7 \cdot 11 \cdot 13)\,(3 \cdot 5 \cdot 9) \\ &= \log_{1001} 1001 \cdot 135 \\ &= \log_{1001} 1001 + \log_{1001} 135 \\ &= 1 + \log_{1001} 135 \end{aligned}$$

and $(b, M) = \boxed{(1001, 135)}$.

Round 5 - Trigonometry

1. Find $\csc\theta$ if $\cos\theta = \frac{\sqrt{3}}{2}$ and $\cot\theta = -\sqrt{3}$.

Solution: Note that $\csc\theta = \dfrac{1}{\sin\theta}$ and that $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{1}{\frac{\sin\theta}{\cos\theta}} = \dfrac{\cos\theta}{\sin\theta}$. Therefore:

$$\begin{aligned}\csc\theta &= \frac{1}{\sin\theta} \\ &= \frac{\cos\theta}{\cos\theta\sin\theta} \\ &= \frac{\cos\theta}{\sin\theta}\frac{1}{\cos\theta} \\ &= \cot\theta\cdot\frac{1}{\cos\theta} \\ &= -\sqrt{3}\cdot\frac{2}{\sqrt{3}} = \boxed{-2}\end{aligned}$$

2. If $\sin A = -\frac{12}{13}$ and $\cos B = -\frac{3}{5}$, where $180^\circ < A < 270^\circ$ and $180^\circ < B < 270^\circ$, find $\cos(A+B)$. Express your result as a fraction in simplest terms.

Solution: First, note that both A and B lie in the third quadrant, so that $\cos A$ and $\sin B$ are both negative. Then the identity $\sin^2\theta + \cos^2\theta = 1$ can be used to find $\cos A$ and $\sin B$:

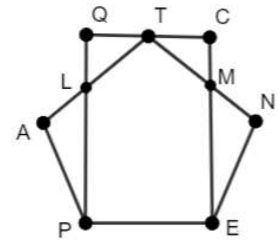
$$\cos A = -\sqrt{1-\sin^2 A} = -\sqrt{1-\frac{12^2}{13^2}} = -\sqrt{1-\frac{144}{169}} = -\sqrt{\frac{169-144}{169}} = -\frac{\sqrt{25}}{\sqrt{169}} = -\frac{5}{13}$$

$$\sin B = -\sqrt{1-\cos^2 B} = -\sqrt{1-\frac{3^2}{5^2}} = -\sqrt{1-\frac{9}{25}} = -\sqrt{\frac{25-9}{25}} = -\frac{\sqrt{16}}{\sqrt{25}} = -\frac{4}{5}$$

Now apply the sum of angles identity for cosine:

$$\begin{aligned}\cos(A+B) &= \cos A\cos B - \sin A\sin B \\ &= \left(-\frac{5}{13}\right)\left(-\frac{3}{5}\right) - \left(-\frac{12}{13}\right)\left(-\frac{4}{5}\right) \\ &= \frac{5\cdot 3}{13\cdot 5} - \frac{12\cdot 4}{13\cdot 5} \\ &= \frac{15-48}{65} \\ &= \boxed{-\frac{33}{65}}.\end{aligned}$$

3. In the figure shown at right, $PENTA$ is a regular pentagon that shares side $PE = 2$ with rectangle $PERQ$. L is the intersection of $\overline{AT}$ and $\overline{PQ}$. Likewise, M is the intersection of $\overline{TN}$ and $\overline{RE}$.
If $AL = \dfrac{a}{b + c\sin^2 18^\circ}$, find (a, b, c).



Solution: First, recall that the measure θ of an the internal angle of a regular polygon with n sides is found by $\theta = \frac{n-2}{n}180^\circ$. For $PENTA$, $n = 5$, and $\theta = \frac{3\cdot 180}{5} = 108^\circ = m\angle APE$. Now $m\angle APL = m\angle APE - m\angle LPE = 108 - 90 = 18^\circ$ because $\angle LPE$ is a right angle. Now AL can be found by the Law of Sines for ΔPAL:

$$\frac{AL}{\sin\angle APL} = \frac{AP}{\sin\angle ALP}$$

Note that $m\angle ALP = 180 - m\angle APL - m\angle PAL = 180 - 18 - 108 = 54^\circ$. Inserting this value and $AP = 2$ into the above proportion:

$$\frac{AL}{\sin 18^\circ} = \frac{2}{\sin 54^\circ}$$

or, noting that $54 = 3 \cdot 18$:

$$AL = \frac{2\sin 18^\circ}{\sin(3 \cdot 18^\circ)}$$

It remains to express $\sin 54^\circ$ in terms of $\sin 18^\circ$ using a triple angle formula. That formula can be derived using the sum of angles and double angle formulas for $\sin A + B$ and $\cos A + B$:

$$\begin{aligned}\sin 3\theta &= \sin(\theta + 2\theta)\\ &= \sin\theta\cos 2\theta + \sin 2\theta\cos\theta\\ &= \sin\theta\left(\cos^2\theta - \sin^2\theta\right) + (2\sin\theta\cos\theta)\cos\theta\\ &= \sin\theta\cos^2\theta - \sin^3\theta + 2\sin\theta\cos^2\theta\\ &= 3\sin\theta\cos^2\theta - \sin^3\theta\\ &= 3\sin\theta\left(1 - \sin^2\theta\right) - \sin^3\theta\\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

Using this formula, $\sin 3 \cdot 18^\circ = 3\sin 18^\circ - 4\sin^3 18^\circ$. AL is found by inserting this expression into the last proportion and simplifying:

$$\begin{aligned}AL &= \frac{2\sin 18^\circ}{3\sin 18^\circ - 4\sin^3 18^\circ}\\ &= \frac{2}{3 - 4\sin^2 18^\circ}\end{aligned}$$

Comparing this expression to the expression in the problem statements, $(a, b, c) = \boxed{(2, 3, -4)}$.

Team Round

1. Three ships travel continuously from New York to Le Havre, France and back. One ship makes the round trip in 12 days, another ship makes the round trip in 20 days, and the third ship makes the round trip in 28 days. If all three ships leave New York on the same day, how many days will elapse before all three ships first arrive in Le Havre on the same day? Assume that the one way trip times are the same for either direction, and that no time is spent in either port.

Solution: The one way trip times for each ship are half the round trip times. Thus, the one way trip times are 6, 10, and 14 days for the three ships. They will all arrive in port on the same day when the number of days that pass is a multiple of 6, 10, and 14. The least common multiple of 6, 10, and 14 is LCM(6,10,14)=LCM$(2\cdot 3, 2\cdot 5, 2\cdot 7) = 2\cdot 3\cdot 5\cdot 7 = 210$. Each ship will be in Le Havre after an odd number of one way trips. Since each ship takes an odd number of one way trips in 210 days $(210 \div 6 = 35, 210 \div 10 = 21, 210 \div 14 = 15)$, all three ships will be in Le Havre $\boxed{210}$ days after they first leave New York.

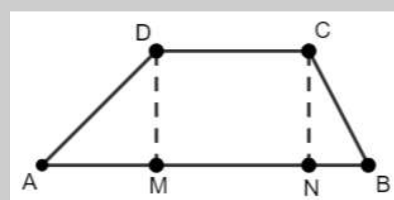
2. In Mr. Smith's math class, 36 students took a test. If the average passing grade was 78, the average failing grade was 60, the class average was 71, and the passing grade was 70, how many students passed the test?

Solution: If P students passed the test, and F students failed the test, then the sum of all passing scores is $78P$, the sum of all failing scores is $60F$, and the sum of all scores, passing and failing, is $78P + 60F = 71(P + F)$. This equation simplifies to $7P = 11F$, so the ratio $P : F = 11 : 7 = 22 : 14$. Since the total number of students is 36, 14 students must have failed and $\boxed{22}$ students passed.

3. Given that $ABCD$ is a trapezoid that has base angles of $\angle DAB$ and $\angle CBA$ measuring 45 degrees and 60 degrees respectively, $BC = 4$ and $DC = 5$, find the area of $ABCD$.

Solution:

Trapezoid $ABCD$ is shown at right. Construct altitudes $\overline{DM}$ and $\overline{CN}$, perpendicular to $\overline{AB}$. ΔCBN is a 30-60-90 triangle whose sides are in proportion $x : x\sqrt{3} : 2x = BN : NC : BC$. Since $BC = 2x = 4$, $x = 2$, $BN = 2$, and the height of the trapezoid $CN = 2\sqrt{3}$.



The area of a trapezoid equals its height times half the sum of its bases, or:

$$[ABCD] = \frac{1}{2}\left(CN \cdot (AB + DC)\right).$$

Now $MN = DC = 5$ and $DM = CN = 2\sqrt{3}$ because $MNCD$ is a rectangle. ΔADM is a 45-45-90 triangle, so $AM = DM = 2\sqrt{3}$. Therefore base $AB = AM + MN + BN = 2\sqrt{3} + 5 + 2 = 7 + 2\sqrt{3}$. Substituting the segment lengths into the equation, the area of $ABCD$ is $\frac{1}{2}\left(2\sqrt{3}(5 + 7 + 2\sqrt{3}\right) = \sqrt{3}(2\sqrt{3} + 12) = \boxed{6 + 12\sqrt{3}}$.

4. The lengths of the sides of a triangle are $\log_3 5$, $\log_3 11$, and $\log_3 x$. If m is the smallest integer value for x, and n is the largest integer value for x, find (m, n).

Solution: Since $\log_3 5$, $\log_3 11$, and $\log_3 x$ are the sides of a triangle, they satisfy the Triangle Inequality: each side length is less than the sum of the other two. Thus:

$$\log_3(11) < \log_3(5) + \log_3(x) = \log_3(5x)$$

and $11 < 5x$ because $\log a < \log b$ implies $a < b$. Thus the smallest possible integer value for x is $m = 3$. Similarly,

$$\log_3 x < \log_3 5 + \log_3 11 = \log_3 5 \cdot 11 = \log_3 55$$

so $\log_3 x < \log_3 55$ and the largest integer value for x is $n = 54$, so $(m, n) = \boxed{(3, 54)}$.

5. In triangle $\triangle ABC$, $AB = 8$, $AC = 20$, and $\tan \angle A = \frac{7}{5}$. Find the area of $\triangle ABC$.

Solution: The area of $\triangle ABC$ is given by the formula $\frac{1}{2} AB \cdot AC \sin \angle A = 80 \sin \angle A$. The sine of an angle can be found from its tangent by constructing a right triangle with the given tangent value, in this case a right triangle whose legs opposite and adjacent to $\angle A$ have lengths 7 and 5, respectively. The hypotenuse of this triangle has length $\sqrt{5^2 + 7^2} = \sqrt{25 + 49} = \sqrt{74}$, and the sine of the angle is equal to the ratio of the opposite length to the hypotenuse, or $\sin(A) = \frac{7}{\sqrt{74}} = \frac{7\sqrt{74}}{74}$. Then the area of the triangle is $80 \cdot \frac{7\sqrt{74}}{74} = \boxed{\dfrac{280\sqrt{74}}{37}}$.

6. Find the polynomial with leading coefficient 1 of lowest order that is a common multiple of $x^2 - 25$, $x^3 - 125$, and $x^2 - 10x + 25$. Provide an answer that is completely factored over the rationals.

Solution: Factoring the individual polynomials:

$$\begin{aligned} x^2 - 25 &= (x-5)(x+5) \\ x^3 - 125 &= (x-5)(x^2 + 5x + 25) \\ x^2 - 10x + 25 &= (x-5)^2 \end{aligned}$$

Thus their least common multiple is $\boxed{(x-5)^2(x+5)(x^2+5x+25)}$.

7. Find all solutions (x, y) to the following system of equations:

$$\frac{x}{4} + \frac{y}{3} = \frac{9}{2}$$
$$\frac{4}{x} + \frac{3}{y} = 1$$

Solution: Let $u = \dfrac{x}{4}$ and $v = \dfrac{y}{3}$. Then the original system is transformed into the following system:

$$u + v = \frac{9}{2}$$
$$\frac{1}{u} + \frac{1}{v} = 1$$

Due to the symmetry of this system u and v will have the same solutions. Isolate u in the second equation, $\dfrac{1}{u} = 1 - \dfrac{1}{v} = \dfrac{v-1}{v}$, then equate the reciprocals of both sides:

$$u = \frac{v}{v-1}.$$

Now substitute this expression for u into the first equation of the transformed system and solve for v:

$$\frac{v}{v-1} + v = \frac{v + v^2 - v}{v-1} = \frac{v^2}{v-1} = \frac{9}{2}$$
$$2v^2 = 9\,(v-1) = 9v - 9$$
$$2v^2 - 9v + 9 = 0$$
$$(2v-3)\,(v-3) = 0$$

where the second line follows by cross multiplying the equation of rational functions. The resulting quadratic is factored in the last line with solutions $v \in \{\frac{3}{2}, 3\}$. Note that $u = 3$ when $v = \frac{3}{2}$ and $u = \frac{3}{2}$ when $v = 3$. Thus the two solutions are $(u, v) \in \left\{(\frac{3}{2}, 3), (3, \frac{3}{2})\right\}$. Finally, $x = 4u$ and $y = 3v$, so the solutions for x and y are $(x, y) \in \left\{\boxed{(6, 9), (12, \frac{9}{2})}\right\}$.

8. Let r be the result of doubling both the base and the exponent of the expression a^b, where both $a > 0$ and $b > 0$. If r is equal to the product of a^b and x^b, find x in terms of a and b.

Solution: First $r = (2a)^{2b} = 2^{2b}a^{2b} = 4^b(a^2)^b$. So $r/a^b = 4^b(a^2)^b/a^b = 4^b a^b = (4a)^b = x^b$. Thus $x = \boxed{4a}$.

9. If $\tan 2x = \frac{24}{7}$ and $90° < x < 180°$, find $\sec x$.

Solution: Note that $\tan 2x = \frac{\sin 2x}{\cos 2x}$. Construct a right triangle with leg lengths 24 and 7; the length of its hypotenuse is $\sqrt{24^2 + 7^2} = \sqrt{576 + 49} = \sqrt{625} = 25$, and the tangent of the angle opposite the leg of length 24 is $\frac{24}{7} = \tan 2x$. Doubling the given inequality $90° < x < 180°$ yields $180° < 2x < 360°$, and $2x$ lies in either the third or fourth quadrant. Since $\tan 2x$ is positive, $2x$ must lie in the third quadrant, and both $\sin 2x$ and $\cos 2x$ are negative. Now $\cos 2x$ is found from the constructed triangle: $\cos 2x = -\frac{7}{25}$.

Modify the double angle formula for cosines using the identity $\sin^2 x = 1 - \cos^2 x$ to express $\cos 2x$ as a function of $\cos x$:

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = -\frac{7}{25}$$

Solve the last equation for $\cos x$:

$$\begin{aligned} 2\cos^2 x &= 1 - \frac{7}{25} = \frac{18}{25} \\ \cos^2 x &= \frac{9}{25} \\ \cos x &= \sqrt{\frac{9}{25}} = \pm\frac{3}{5} \end{aligned}$$

And $\cos x = -\frac{3}{5}$ because x is in the second quadrant. Finally, $\sec x = \dfrac{1}{\cos x} = \boxed{-\dfrac{5}{3}}$.